

Mathematics Applied to Quantitative Social Sciences

Intermediate Level

Session 9

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Sequences

Definition

A sequence can be thought of as a list of numbers written in a definite order:

$$a_1, a_2, a_3, a_4, \dots, a_n$$

The number a_1 is called the first term, a_2 is the second term, and in general a_n is the n th term. For every positive integer n there is a corresponding number a_n and so a sequence can be defined as a function whose domain is the set of positive integers.

The sequence $\{a_1, a_2, a_3, \dots\}$ is also denoted by:

$$\{a_n\} \text{ or } \{a_n\}_{n=1}^{\infty}$$

Sequences

Sequences can be defined by giving a formula for the n th term, by using the defining formula, or by writing out the terms of the sequence.

For example:

$$\left\{ \frac{n}{n+1} \right\}_{n=1}^{\infty} \quad a_n = \frac{n}{n+1} \quad \left\{ \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \dots \right\}$$

$$\left\{ \frac{(-1)^n(n+1)}{3^n} \right\}_{n=1}^{\infty} \quad a_n = \frac{(-1)^n(n+1)}{3^n} \quad \left\{ -\frac{2}{3}, \frac{3}{9}, -\frac{4}{27}, \dots \right\}$$

$$\left\{ \sqrt{n-3} \right\}_{n=3}^{\infty} \quad a_n = \sqrt{n-3}, \quad n \geq 3 \quad \{0, 1, \sqrt{2}, \sqrt{3}, \dots\}$$

A.1 Find a formula for the general term a_n of the

sequence: $\left\{ \frac{3}{5}, -\frac{4}{25}, \frac{5}{125}, -\frac{6}{625}, \dots \right\}$

Limits of a Sequence

Definition

A sequence $\{a_n\}$ has the limit L and we write:

$$\lim_{n \rightarrow \infty} a_n = L \quad \text{or} \quad a_n \rightarrow L \text{ as } n \rightarrow \infty$$

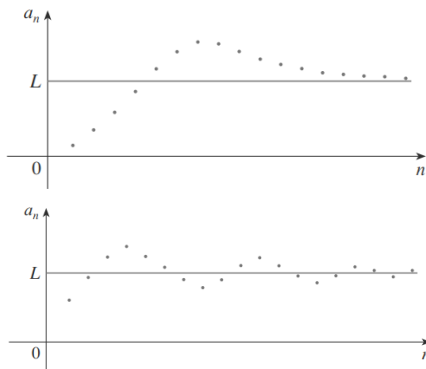
if for every $\varepsilon > 0$ there is a corresponding integer N such that:

$$|a_n - L| < \varepsilon \quad n > N$$

- If $\lim_{n \rightarrow \infty} a_n = L$ exists, we say the sequence **converges**
- If $\lim_{n \rightarrow \infty} a_n = L$ does not exist, we say the sequence **diverges**

Limits of a Sequence

Graphically:



Limit Laws for Sequences

If $\{a_n\}$ and $\{b_n\}$ are convergent sequences and c is a constant, then:

- $\lim_{n \rightarrow \infty} (a_n + b_n) = \lim_{n \rightarrow \infty} a_n + \lim_{n \rightarrow \infty} b_n$
- $\lim_{n \rightarrow \infty} (a_n - b_n) = \lim_{n \rightarrow \infty} a_n - \lim_{n \rightarrow \infty} b_n$
- $\lim_{n \rightarrow \infty} ca_n = c \lim_{n \rightarrow \infty} a_n$ $\lim_{n \rightarrow \infty} c = c$
- $\lim_{n \rightarrow \infty} (a_n b_n) = \lim_{n \rightarrow \infty} a_n \times \lim_{n \rightarrow \infty} b_n$
- $\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \frac{\lim_{n \rightarrow \infty} a_n}{\lim_{n \rightarrow \infty} b_n}$ if $\lim_{n \rightarrow \infty} b_n \neq 0$
- $\lim_{n \rightarrow \infty} a_n^p = \left[\lim_{n \rightarrow \infty} a_n \right]^p$ if $p > 0$ and $a_n > 0$

B.1 Find $\lim_{n \rightarrow \infty} \frac{n}{n+1}$

The Squeeze Theorem

Given $\{a_n\}$ and $\{c_n\}$, convergent sequences.

If

$$a_n \leq b_n \leq c_n \text{ for } n \geq n_0$$

and

$$\lim_{n \rightarrow \infty} a_n = L = \lim_{n \rightarrow \infty} c_n$$

Then

$$\lim_{n \rightarrow \infty} b_n = L$$

Absolute Value Theorem

Given the sequence $\{a_n\}$

If

$$\lim_{n \rightarrow \infty} |a_n| = 0$$

Then

$$\lim_{n \rightarrow \infty} a_n = 0$$

C.1 Evaluate $\lim_{n \rightarrow \infty} \frac{(-1)^n}{n}$

Monotonic Sequences

A sequence is Monotonic if it is either increasing or decreasing.

- It is called **increasing** if $a_n < a_{n+1}$ for all $n \geq 1$
- It is called **decreasing** if $a_n > a_{n+1}$ for all $n \geq 1$

Example

$\left\{ \frac{3}{n+5} \right\}$ is decreasing because:

$$\frac{3}{n+5} > \frac{3}{(n+1)+5} = \frac{3}{n+6}$$

For all $n \geq 1$

C.2 Determine if $a_n = \frac{n}{n^2+1}$ is decreasing.

Bounded Sequences

Definition

- A sequence $\{a_n\}$ is **bounded above** if there is a number M such that

$$a_n \leq M \quad \text{for all } n \geq 1$$

- A sequence $\{a_n\}$ is **bounded below** if there is a number m such that

$$m \leq a_n \quad \text{for all } n \geq 1$$

If it is bounded above and below, then $\{a_n\}$ is a **bounded sequence**.

Example

$\left\{ \frac{n}{n+1} \right\}$ is bounded because $0 < a_n < 1$ for all n .

Every bounded, monotonic sequence is convergent

Series

Definition

If we try to add the terms of an infinite sequence $\{a_n\}_{n=1}^{\infty}$ we get an expression of the form:

$$a_1 + a_2 + a_3 + a_4 + \cdots + a_n + \cdots$$

which is called an **infinite series** (or just a series) and is denoted, for short, by the symbol:

$$\sum_{n=1}^{\infty} a_n \quad \text{or} \quad \sum a_n$$

Partial Sums

Representation

Partial sums are given by:

$$S_1 = a_1$$

$$S_2 = a_1 + a_2$$

$$S_3 = a_1 + a_2 + a_3$$

And in general $S_n = a_1 + a_2 + a_3 + \cdots + a_n = \sum_{i=1}^n a_i$

These partial sums form a new sequence $\{S_n\}_{n=1}^{\infty}$, which may or may not have a limit.

Convergent and Divergent Series

Definition

Given a series $\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \dots$, let s_n denote its n th partial sum:

$$S_n = a_1 + a_2 + a_3 + \dots + a_n = \sum_{i=1}^n a_i$$

If the sequence $\{S_n\}$ is convergent and $\lim_{n \rightarrow \infty} S_n = S$ exists as a real number, then the series $\sum a_n$ is called convergent and we write

$$a_1 + a_2 + \dots + a_n + \dots \quad \text{or} \quad \sum_{n=1}^{\infty} a_n = S$$

The number S is called the **sum** of the series. Otherwise, the series is called **divergent**

Convergent and Divergent Series

Thus, when we write $\sum_{n=1}^{\infty} a_n = s$ we mean that by adding sufficiently many terms of the series we can get as close as we like to the number S . Notice that

$$\sum_{n=1}^{\infty} a_n = \lim_{n \rightarrow \infty} \sum_{i=1}^n a_i$$

Geometric Series

In this series, each term is obtained from the preceding one by multiplying it by the common ratio r .

Sum of a Geometric series

The Geometric series

$$\sum_{n=1}^{\infty} ar^{n-1} = a + ar + ar^2 + \dots$$

is convergent if $|r| < 1$ and its sum is

$$\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r}$$

If , $|r| \geq 1$, the geometric series is divergent.

Geometric Series

Find the sum of the following series:

D.1 $5 - \frac{10}{3} + \frac{20}{9} - \frac{40}{27} + \dots$

D.2 $\sum_{n=1}^{\infty} 2^{2n} 3^{1-n}$

Properties of Convergent Series

Properties

If $\sum a_n$ and $\sum b_n$ are convergent series, and c is a constant:

- $\sum_{n=1}^{\infty} ca_n = c \sum_{n=1}^{\infty} a_n$
- $\sum_{n=1}^{\infty} (a_n + b_n) = \sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} b_n$
- $\sum_{n=1}^{\infty} (a_n - b_n) = \sum_{n=1}^{\infty} a_n - \sum_{n=1}^{\infty} b_n$

Convergency and Divergency

Convergency Condition

If the series $\sum_{n=1}^{\infty} a_n$ is convergent, then $\lim_{n \rightarrow \infty} a_n = 0$

Divergency Test

If $\lim_{n \rightarrow \infty} a_n$ does not exist or if $\lim_{n \rightarrow \infty} a_n \neq 0$, then the series $\sum_{n=1}^{\infty} a_n$ is divergent.

D.3 Determine whether $\sum_{n=1}^{\infty} \frac{n^2}{5n^2 + 4}$ is convergent or divergent