

Mathematics Applied to Quantitative Social Sciences

Intermediate Level

Session 8

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Antiderivatives

Definition

A function F is called an **antiderivative** of a function f on an interval I if $F'(x) = f(x)$ for all x in I .

Theorem

If F is an antiderivative of f on an interval I , then the most general antiderivative of f on I is

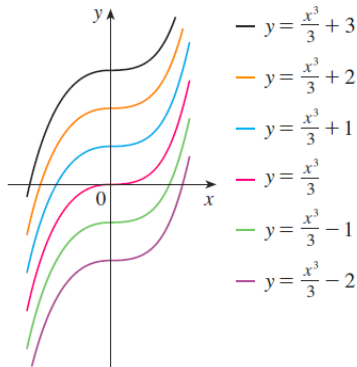
$$F(x) + c$$

where C is an arbitrary constant.

- Example : An antiderivative of $f(x) = x^2$

Antiderivatives

The general derivative of $f(x) = x^2$ is $\frac{1}{3}x^3 + C$



Each of these curves must have the same slope at any given value of x .

Some Obvious Antiderivatives

If $f(x) = \dots$, then $F(x) = \dots$ is an antiderivative of f :

$f(x)$	$F(x)$
$x^n \ (n \neq -1)$	$\frac{x^{n+1}}{n+1}$

e^x	e^x
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$\frac{1}{x}$	$\ln x $
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$\frac{1}{\sqrt{x}}$	$2\sqrt{x}$
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$f(x)$	$F(x)$
$u'(x)u(x)^n \ (n \neq -1)$	$\frac{u(x)^{n+1}}{n+1}$

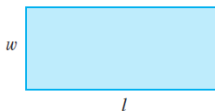
$u'(x)e^{u(x)}$	$e^{u(x)}$
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$\frac{u'(x)}{u(x)}$	$\ln u(x) $
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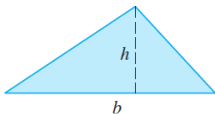
$\frac{u'(x)}{\sqrt{u(x)}}$	$2\sqrt{u(x)}$
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Calculating Simple Areas

- Area of rectangle: $A = w \times l$

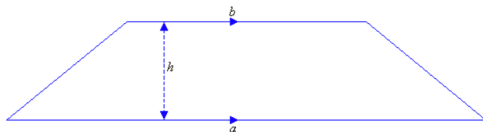


- Area of a triangle : $A = \frac{b \times h}{2}$

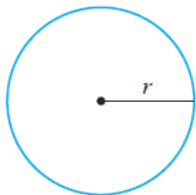


Calculating Simple Areas

- Area of a trapezium: $A = h \times \frac{a + b}{2}$

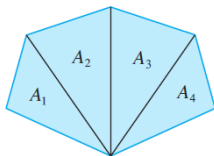


- Area of circle : $A = \pi \times r^2$

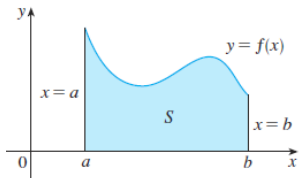


Calculating (less simple) Areas

- Area of a polygon : divide the polygon into triangles and add the areas of the triangles: $A = A_1 + A_2 + A_3 + A_4$



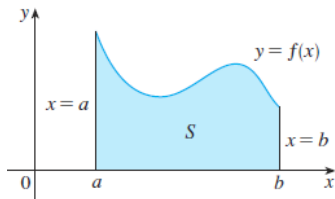
- What about the area of a region with curved sides?



The Area Problem

The Area Problem

Find the area of the region S bounded by the graph of a continuous positive function f , the vertical lines $x = a$ and $x = b$, and the x -axis.

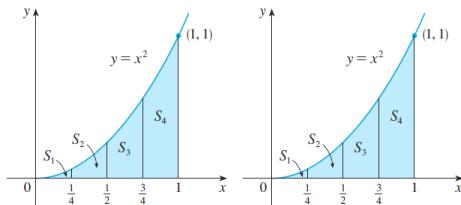


- Idea: approximate the region S with rectangles, increase the number of rectangles, take the limit of the areas of the rectangles.

The Area Problem

Example

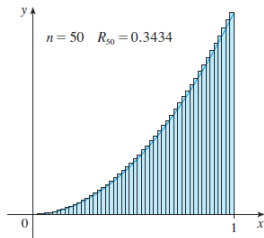
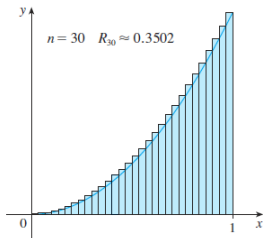
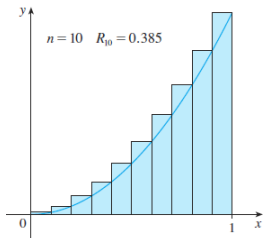
- Area under the parabola $y = x^2$ from 0 to 1
- We divide S into four strips and approximate each strip by a rectangle



R_4 = the sum of the areas of the 4 rectangles

$$\begin{aligned} &= \frac{1}{4} \cdot \left(\frac{1}{4}\right)^2 + \frac{1}{4} \cdot \left(\frac{1}{2}\right)^2 + \frac{1}{4} \cdot \left(\frac{3}{4}\right)^2 + \frac{1}{4} \cdot 1^2 \\ &= \frac{15}{32} = 0.46875 \end{aligned}$$

The Area Problem



If we keep increasing the number n of rectangles, R_n will get close to a particular value (0.33333... in this case).

We define the area A to be the limit of the sums of the areas of the approximating rectangles :

$$A = \lim_{n \rightarrow +\infty} R_n = \frac{1}{3}$$

The Area Problem

Definition

The area of the region S that lies under the graph of the continuous function f is the limit of the sum R_n of the areas of the n approximating rectangles :

$$A = \lim_{n \rightarrow \infty} R_n$$

The previous limit, if it exists, is called the **definite integral** of f from a to b :

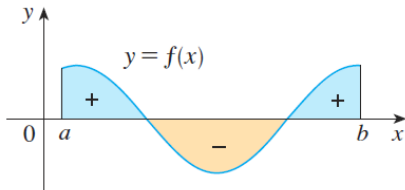
$$\int_a^b f(x) dx = \lim_{n \rightarrow \infty} R_n$$

If it does exist, we say that f is **integrable** on $[a, b]$.

The previous sum R_n is called a **Riemann sum**.

The Definite Integral

If f takes on both positive and negative values:



Interpretation of the definite integral

$$\int_a^b f(x) dx = A_1 - A_2 = \text{The net area}$$

where

A_1 = The area of the blue region

A_2 = The area of the light brown region

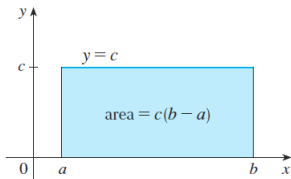
The Definite Integral

Properties

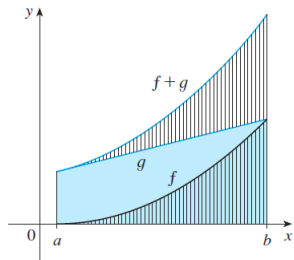
- $\int_a^b f(x) dx = - \int_b^a f(x) dx$
- $\int_a^a f(x) dx = 0$
- $\int_a^b c dx = c(b - a)$ where c is any constant
- $\int_a^b [f(x) + g(x)] dx = \int_a^b f(x) dx + \int_a^b g(x) dx$
- $\int_a^b [f(x) - g(x)] dx = \int_a^b f(x) dx - \int_a^b g(x) dx$
- $\int_a^b c f(x) dx = c \int_a^b f(x) dx$

The Definite Integral

Some intuitive proofs:



$$\int_a^b c \, dx = c(b-a)$$

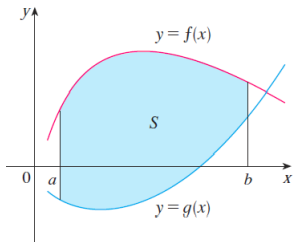
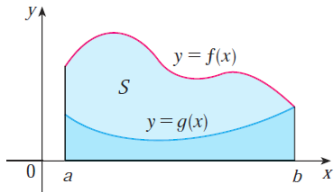


$$\int_a^b [f(x) + g(x)] \, dx = \int_a^b f(x) \, dx + \int_a^b g(x) \, dx$$

Areas Between Curves

The area A of the region S bounded by the curves $y = f(x)$ and $y = g(x)$ and the lines $x = a$ and $x = b$, where f and g are continuous and $f(x) \geq g(x)$ for all $x \in [a, b]$ is:

$$A = \int_a^b [f(x) - g(x)] dx$$



The fundamental theorem of Calculus

How are definite integrals and antiderivatives connected ?

Theorem

Suppose that f is continuous on $[a, b]$, we have:

$$\int_a^b f(x) dx = F(b) - F(a)$$

where F is any antiderivative of f , that is, a function such that $F' = f$.

$F(b) - F(a)$ is denoted $F(X) \Big|_a^b$

Examples:

$$\bullet \int_1^3 e^x dx = e^x \Big|_1^3 = e^3 - e \quad \bullet \int_3^6 \frac{1}{x} dx = \ln(x) \Big|_3^6 = \ln(6) - \ln(3) = \ln(2)$$

Examples

Evaluate the following integrals :

$$\text{A.1 } \int_1^2 (8x^3 + 3x) dx$$

$$\text{A.2 } \int_0^1 (1 - x^9) dx$$

$$\text{A.3 } \int_1^9 \frac{\sqrt{x} - 2x^2}{x} dx$$

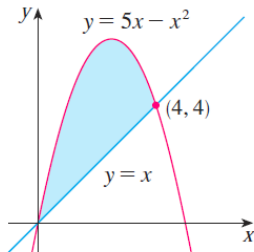
$$\text{A.4 } \int_0^1 y(1 + y^2)^5 dy$$

$$\text{A.5 } \int_0^1 (1 - x)^9 dx$$

$$\text{A.6 } \int_{-4}^4 2xe^{x^2} dx$$

$$\text{A.7 } \int_e^{e^2} \frac{1}{x \ln(x)} dx$$

A.8 Find the area of the shaded region



Application to Integration

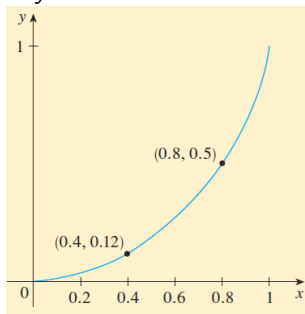
The **Lorenz curve** is a graph of income inequality that shows what percentage of a country's income or wealth is being earned by a percentage of the country's households or population.

Properties

A **Lorenz curve** is the graph of a function f having the following properties :

- f is defined, continuous and increasing on $[0, 1]$.
- for all $x \in [0, 1]$, $0 \leq f(x) \leq x$ (f is under the line $y = x$)
- $f(0) = 0$ and $f(1) = 1$

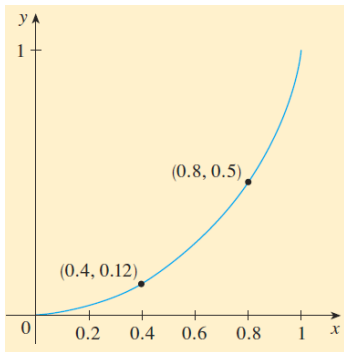
$x =$ Population fraction
 $y =$ Income fraction



Application to Integration

Lorenz curves and Gini coefficient

How to read a Lorenz Curve?



- 40% of the population received just 12% of the total income.
- 80% of the population received 50% of the total income.
- The greater the inequality, the more the line curves away from the diagonal $y = x$

x = Population fraction

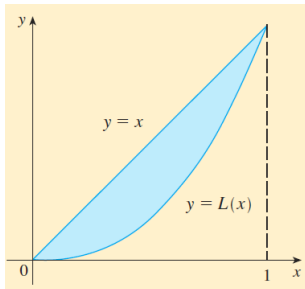
y = Income fraction

Application to Integration

Lorenz curves and Gini coefficient

Gini index

The **Gini coefficient** G (or Gini index) is the area between the Lorenz curve L and the line $y = x$ (shaded region) divided by the area under the line $y = x$.



$$G = \frac{\text{area of the shaded region}}{1/2}$$

$$G = 2 \times (\text{area of the shaded region})$$

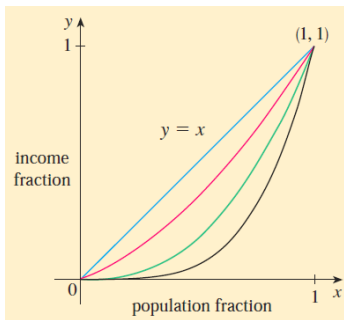
Gini Index

$$G = 2 \int_0^1 [x - L(x)] dx$$

Application to Integration

Lorenz curves and Gini coefficient

Increasing Gini coefficient \Leftrightarrow Increasing the wealth inequality



Extreme cases

- **perfectly egalitarian society** : everybody has the same income.
- **perfectly totalitarian society** : a single person receives all the income.

What is Lorenz curve and Gini coefficient in both cases ?