Mathematics Applied to Quantitative Social Sciences
Intermediate Level

Session 7

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Optimization Problem

- Input description : a function $f(x_1, \ldots, x_n)$ of n variables.
- Problem description : what point $a = (a_1, \ldots, a_n)$ maximizes (or minimizes) the function f?
  - maximum of f = minimum of $-f$
- Constrained or unconstrained optimization?
  - **Unconstrained optimization :** No limitations on the values of the variables.
  - **Constrained optimization :** The variables are not free to take on any value but are constrained

Example

Utility Maximization Problem

- Consider n commodities, enumerated $1, \cdots, n$.
- Let x_i represent the amount of commodity i and $U(x_1, \cdots, x_n)$ be the function measuring the individual's level of utility or satisfaction when consuming x_1 units of good 1, x_2 units of good 2, and so on.
- Let $p_1, \cdots, p_n$ denote the prices of the n commodities.
- Let I denote the individual's income.

The consumer wants to maximize

$$U(x_1, \cdots, x_n)$$
$$\text{subject to} \quad p_1 x_1 + \cdots + p_n x_n \leq I$$
$$\text{and} \quad x_1 \geq 0, \cdots, x_n \geq 0$$

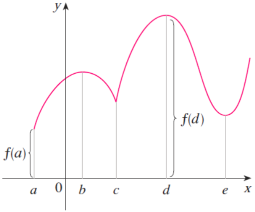
Definitions

Let $f : D \longrightarrow \mathbb{R}$ be a real-valued function of n variables, whose domain D is a subset of $\mathbb{R}^n$. We say that the function f.

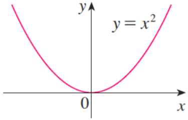
- has a global (or absolute) maximum at a point $a \in D$ if $f(a) \geq f(x)$ for all $x \in D$.
- has a global (or absolute) minimum at a point $a \in D$ if $f(a) \leq f(x)$ for all $x \in D$.
- has a local maximum at a point $a \in D$ if $f(a) \geq f(x)$ for every x near a.
- has a local minimum at a point $a \in D$ if $f(a) \leq f(x)$ for every x near a.

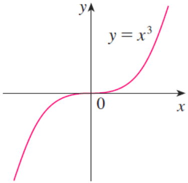
Examples

- "near a" = in an open interval containing a.



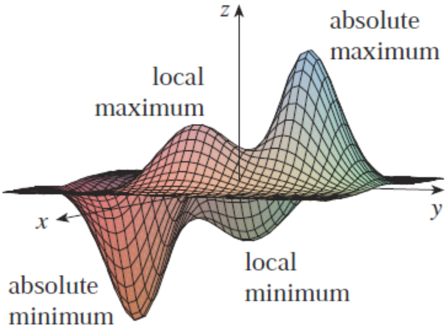
A.1 Find the global and local max and min of the following functions





Examples

- "near a" = in an open disk containing a.



Unconstrained Optimization

Conditions for local minimum/maximum

	One Variable Functions	Two Variable Functions
First Order Conditions	Involves f'	Involves f_x, f_y
Second Order Conditions	Involves f''	Involves f_{xx}, f_{xy}, f_{yx} and f_{yy}

One variable functions

Local extrema - First order condition

- A function f is said to be of class C^1 if f' exists and f' is continuous.
- A function f is said to be of class C^2 if f'' exists and f'' is continuous.

First order condition

Let $f : I \longrightarrow \mathbb{R}$ be a C^1 function defined on an open interval I of $\mathbb{R}$. If f has a local minimum or maximum at a point $c \in I$, then

$$f'(c) = 0$$

Such points are called critical points of f.

This is a **necessary condition** for a local minimum or maximum: condition satisfied by every local minimum or maximum

One variable functions

Local extrema - Second order condition

Second order condition

Let $f : I \longrightarrow \mathbb{R}$ be a C^2 function defined on an open interval I of $\mathbb{R}$.
Suppose that c is a point in I such that $f'(c) = 0$

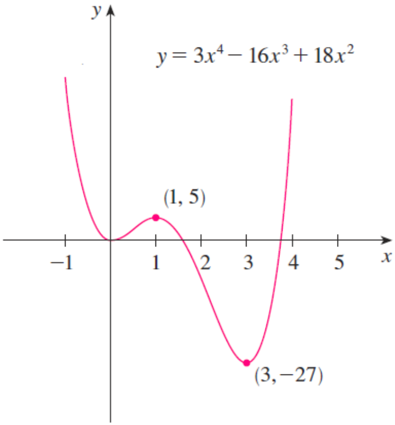
- If $f''(c) > 0$, then f has a **local minimum** at c.
- If $f''(c) < 0$, then f has a **local maximum** at c.

This is a **sufficient condition** for a local minimum or maximum : condition which guarantees a local minimum or maximum

- If $f''(c) = 0$, the test gives no information.

Example

Find the local maximum and minimum of
B.1 $3x^4 - 16x^3 + 18x^2$ on [-1,4]



Functions of two variables

Gradient

The Gradient Vector

If f is a function of two variables x and y, then the gradient of f is the vector $\vec{\nabla} f$ defined by

$$\vec{\nabla} f(x,y) = \begin{pmatrix} f_x(x,y) \\ f_y(x,y) \end{pmatrix}$$

- Gradients can be defined for functions of more than two variables. For instance, if f is a function of three variables x, y, z, then

$$\vec{\nabla} f(x,y,z) = \begin{pmatrix} f_x(x,y,z) \\ f_y(x,y,z) \\ f_z(x,y,z) \end{pmatrix}$$

Example

C.1 Find the gradient of the following functions, then evaluate the gradient at the given point P :

- $f(x, y) = \dfrac{x}{(x^2 + y^2)}$, at $P(-6, 4)$
- $g(x, y) = y\sqrt{x}$, at $P(4, 1)$
- $h(x, y, z) = x^2yz - xyz^3$, at $P(2, -1, 1)$
- $F(x, y, z) = x\mathrm{e}^y + y\mathrm{e}^z + z\mathrm{e}^x$, at $P(0, 0, 0)$

Two variables functions

Local extrema - First order condition

First order condition

Let $f : U \longrightarrow \mathbb{R}$ be a C^1 function defined on an open subset U of $\mathbb{R}^2$. If f has a local minimum or maximum at a point $(a, b) \in U$, then

$$\vec{\nabla} f(x, y) = \vec{0}$$

Such points (a, b) are called **critical points** of f.

This is a **necessary condition** for a local minimum or maximum condition satisfied by every local min or max.

C.2 Find the critical points of the following functions:

- $f(x, y) = x^2 + y^2 - 2x - 6y + 7$
- $g(x, y) = x^2 - y^2 - xy - 4y - x + 16$
- $h(x, y) = x^4 + y^4 - 4xy + 1$

Two variable functions

Local extrema - Second order condition

Second order condition

Let $f : U \longrightarrow \mathbb{R}$ be a C^2 function defined on an open set U of $\mathbb{R}^2$. Suppose that (a, b) is a point in U **such that** $\vec{\nabla} f(x, y) = \vec{0}$. Let

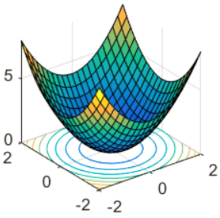
$$D = f_{xx}(a, b) f_{yy}(a, b) - [f_{xy}(a, b)]^2$$

- If $D > 0$, and $f_{xx} > 0$, then f has a **local minimum** at (a, b).
- If $D > 0$, and $f_{xx} < 0$, then f has a **local maximum** at (a, b).
- If $D < 0$, then f has neither a local minimum nor a local maximum at (a, b): we say that (a, b) is a **saddle point**.

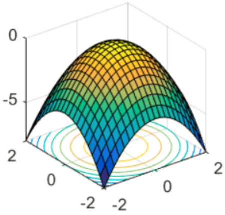
If $D = 0$, the test gives no information.

Example

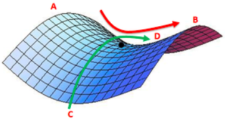
Local Minimum



Local Maximum



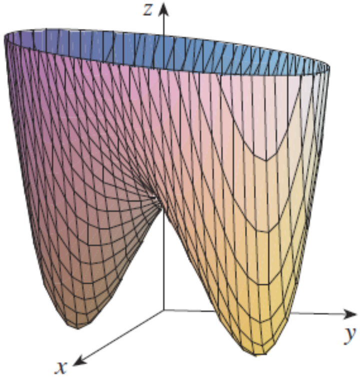
Saddle Point



Example

Find the local minimum, local maximum and saddle points of

D.1 $f(x, y) = x^4 + y^4 - 4xy + 1$



Summary

Unconstrained Optimization

Conditions for local minimum/maximum

	One Variable Functions	Two Variable Functions
First Order Conditions	Solve $f'(x) = 0$ to find the critical points	Solve $\vec{\nabla} f(x, y) = \vec{0}$ to find the critical points
Second Order Conditions	check the sign of f'' at the critical points	check the sign of D at the critical points

Constrained Optimization

Two variables and one equality constraint - Substitution method

Problem description : maximize the objective function $f(x, y)$ subject to the constraint $\varphi(x, y) = 0$

- Use the constraint equation $\varphi(x, y) = 0$ to express one of the variables in terms of the other. For example, y in terms of x : we get $y = h(x)$
- Use this equation to eliminate y from the objective function $f(x, y)$, so that $f(x, y) = f(x, h(x)) = g(x)$
- Now we are left with the unconstrained maximization problem of $g(x)$: we can apply tools developed for unconstrained optimization of functions of one variable.
- Substituting x back into the constraint equation yields a value for y.

Examples

E.1 Minimize $f(x, y) = x^2 + y^2$ subject to $x + y - 1 = 0$

- Use the equation $x + y - 1 = 0$ to express y in terms of x, so that $y = 1 - x = h(x)$.
- Eliminate y from $f(x, y)$ to get a function $g(x) : f(x, y) = f(x, 1 - x) = x^2 + (1 - x)^2 = 2x^2 - 2x + 1$

$$g(x) = 2x^2 - 2x + 1$$

- Minimize $g(x)$:
  - Critical points: $g'(x) = 0 \Leftrightarrow x = \dfrac{1}{2}$
  - $g''(\dfrac{1}{2}) = 4 > 0 \rightsquigarrow g$ has a local minimum at $= \frac{1}{2}$
- Under the constraint $x + y - 1 = 0$, $f(x, y)$ has a local minimum at $\left(\dfrac{1}{2}, h\left(\dfrac{1}{2}\right)\right) = \left(\dfrac{1}{2}, \dfrac{1}{2}\right)$

Examples

E.2 Consider the utility function $U(x,y) = \sqrt{x} + \sqrt{y}$ to be maximized subject to the budget constraint $100 = 10x + 5y$ (Good A costs 10\$/unit and good B costs 5\$/unit. The consumer has 100 dollars to spend and is buying x units of good A and y units of good B).

- From the constraint equation, we get $y = 20 - 2x = h(x)$.
- Substituting into the objective function, it yields a new function entirely in terms of x :

$$g(x) = \sqrt{x} + \sqrt{20 - 2x}$$

- We maximize $g(x)$:
  - Critical points: $g'(x) = 0 \Leftrightarrow x = \frac{10}{3}$
  - $g''\left(\frac{10}{3}\right) < 0 \rightsquigarrow g$ has a local maximum at $x = \frac{10}{3}$
- Under the constraint $10x + 5y = 100$, $f(x,y)$ has a local maximum at $\left(\frac{10}{3}, h\left(\frac{10}{3}\right)\right) = \left(\frac{10}{3}, \frac{40}{3}\right) = (3.33, 13.33)$