Mathematics Applied to Quantitative Social Sciences Intermediate Level

Session 6

Juliana Pinillos
juliana.pinillos@sciencespo.fr

Sciences Po

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Contents

Applications of Differentiation

Theorem

If $f'(x) = 0$ for all x in an interval $[a, b]$, then f is constant on $[a, b]$

Corollary

If $f'(x) = g'(x)$ for all x in an interval $[a, b]$, then the function $f - g$ is constant on $[a, b]$

Definition

A **critical number** of a function f is a number c in the domain of f such that either:

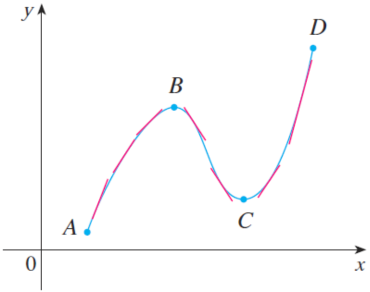
$$f'(c) = 0 \quad \text{OR} \quad f'(c) \textbf{ does not exist}$$

What Does f' Say About f ?

Increasing/Decreasing Test

- If $f'(x) > 0$ on an interval, then f is increasing on that interval.
- If $f'(x) < 0$ on an interval, then f is decreasing on that interval.

Intuitive Proof:



What Does f' Say About f ?

The First Derivative Test

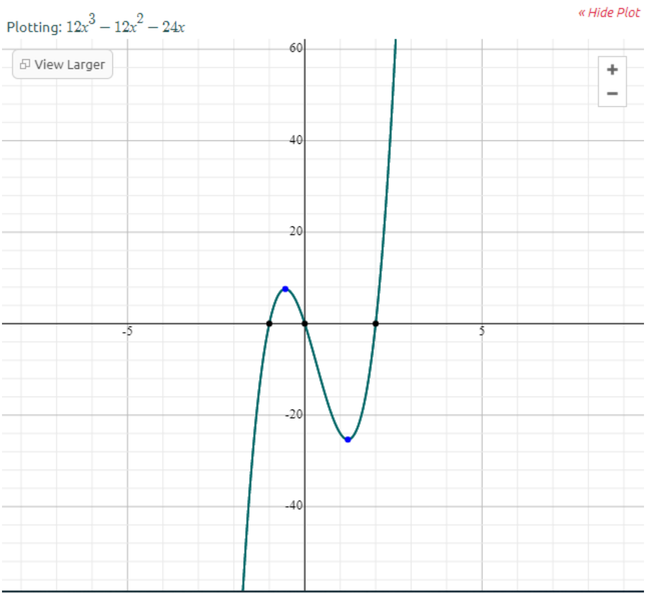
Suppose that c is a critical number of a continuous function f

- If f' changes from positive to negative at c, then f has a local maximum at c.
- If f' changes from negative to positive at c, then f has a local minimum at c.
- If f' does not change sign at c, then f has no local maximum or minimum at c.

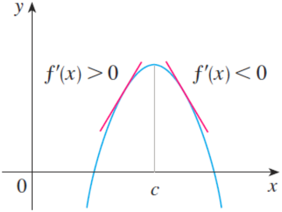
Proof : consequence of the Increasing/Decreasing test

A.1 Find where the function $f(x) = 3x^4 - 4x^3 - 12x^2 + 5$ is increasing and where it is decreasing

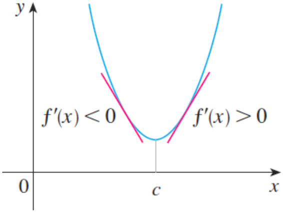
Plotting: $12x^3 - 12x^2 - 24x$



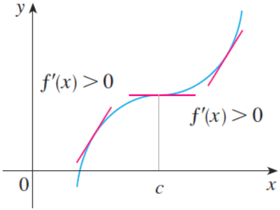
What Does f' Say About f ?



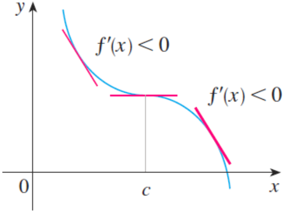
(a) Local maximum



(b) Local minimum



(c) No maximum or minimum

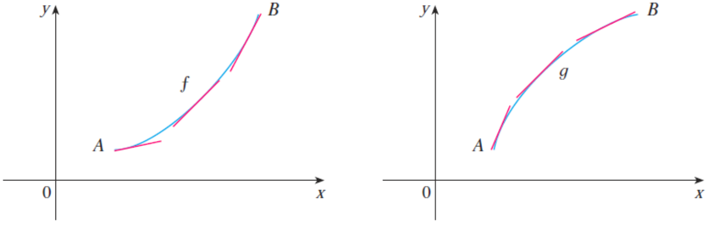


(d) No maximum or minimum

What Does f'' Say About f ?

Definition

- If the graph of f lies above all of its tangents on an interval I, then it is called **concave upward/Convex** on I.
- If the graph of f lies below all of its tangents on an interval I, then it is called **concave downward** on I.



(a) Concave upward (b) Concave downward

What Does f'' Say About f ?

Concavity Test

- If $f''(x) > 0$ for all x in I, then the graph of f is concave upward on I.
- If $f''(x) < 0$ for all x in I, then the graph of f is concave downward on I.

Intuitive proof:

$f''(x) > 0 \Rightarrow f'$ is increasing $\Rightarrow$ the slope of the tangent line is increasing $\Rightarrow$ the graph of f is above its tangents $\Rightarrow f$ is CU

$f''(x) < 0 \Rightarrow f'$ is decreasing $\Rightarrow$ the slope of the tangent line is decreasing $\Rightarrow$ the graph of f is above its tangents $\Rightarrow f$ is CD

What Does f'' Say About f ?

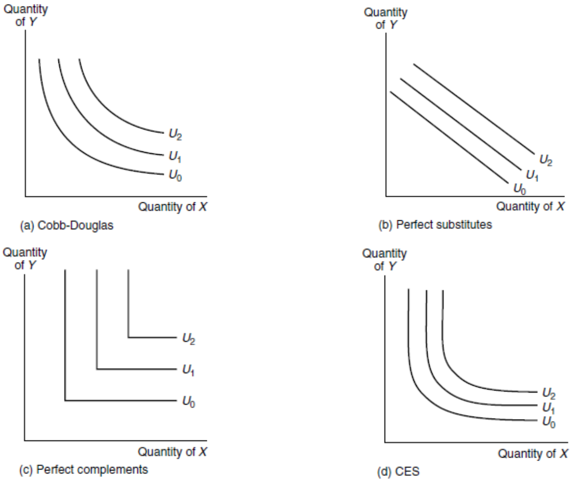
Definition

A point P on a curve $u = f(x)$ is called an inflection point if f is continuous there and the curve changes from CU to CD or from CD to CU at P.

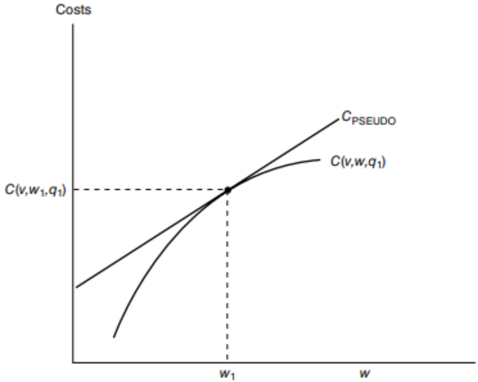
B.1 Sketch a possible graph of a function f that satisfies the following conditions :

- $f'(x) > 0$ on $[-\infty, 1]$ and $f'(x) < 0$ on $[1, +\infty]$
- $f''(x) > 0$ on $[-\infty, -2] \cup [2, \infty]$ and $f''(x) < 0$ on $[-2, 2]$
- $\lim_{x \to -\infty} f(x) = -2$ and $\lim_{x \to +\infty} f(x) = 2$

Examples of Utility Functions



Example of a Cost Function



Minimizing Costs

C.1 A manufacturer estimates that if x units of a particular commodity are produced, the total cost will be C(x) dollars, where

$$C(x) = x^3 - 24x^2 + 350x + 338$$

- At what level of production will the marginal cost $C'(x)$ be minimized?
- Find an expression for the average cost, and find it's derivative.

Maximizing Profits

C.2 A manufacturer estimates that when q thousand units of a particular commodity are produced each month, the total cost will be $C(q) = 0.4q^2 + 3q + 40$ thousand dollars, and all q units can be sold at a price of $p(q) = 22.2 - 1.2q$ dollars per unit. Determine the level of production that results in maximum profit. What is the maximum profit?

Partial Derivatives

Definition

If f is a function of two variables, its partial derivatives are the functions f_x and f_y, also denoted by $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$, defined by

$$f_x(a,b) = \frac{\partial f}{\partial x}(a,b) = \lim h \to 0 \frac{f(a+h,b) - f(a,b)}{h}$$

$$f_y(a,b) = \frac{\partial f}{\partial y}(a,b) = \lim h \to 0 \frac{f(a,b+h) - f(a,b)}{h}$$

The partial derivatives may also be denoted by $D_x f$ and $D_y f$.

Computing Partial Derivatives

...what we need to know :

Rules for finding partial derivatives of $f(x, y)$

- To find f_x, consider y as a constant and differentiate $f(x, y)$ with respect to x.
- To find f_y, consider x as a constant and differentiate $f(x, y)$ with respect to y.

D.1 Find the partial derivatives of $g(x, y) = \frac{x}{y} - \frac{y}{x}$

D.2 If $f(x, y) = x^3 + x^2y^3 - 2y^4$

- $f_x(2, 1) = ?$
- $f_y(2, 1) = ?$

Functions of more than two variables

- Partial derivatives can also be defined for functions of three variables or more.
- For example, if f is a function of three variables x, y and z, then its partial derivative with respect to x is found by regarding y and z as constants and differentiating $f(x, y, z)$ with respect to x.
- Examples :

  E.1 If $f(x, y, z) = \mathrm{e}^{xy} \ln(z)$ find f_x, f_y and f_z

  E.2 If $g(x, y, z) = xyz \ln(x + y + z)$ find g_x, g_y and g_z

Second Partial Derivatives

If f is a function of two variables, then its partial derivatives f_x and f_y are also functions of two variables and so we can consider their partial derivatives $(f_x)_x$ $(f_x)_y$, $(f_y)_x$ and $(f_y)_y$, which are called the **second partial derivatives** of f.

Notation

- $(f_x)_x = f_{xx} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2}$
- $(f_x)_y = f_{xy} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y \partial x}$
- $(f_y)_x = f_{yx} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x \partial y}$
- $(f_y)_y = f_{yy} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial y^2}$

Second Partial Derivatives

Theorem

Suppose f is defined on a disk D that contains the point (a, b). If f_{xy} and f_{yx} are both continous on D, then.

$$f_{xy}(a, b) = f_{yx}(a, b)$$

Example: If $f(x, y) = x^3 + x^2y^3 - 2y^2 + \mathrm{e}^{x^2y}$

F.1 $f_{xx} =?$

F.2 $f_{xy} =?$

F.3 $f_{yx} =?$

F.4 $f_{yy} =?$

Marginal Utility

Marginal Utility

Given the utility function of an individual

$$Utility = U(X_1, X_2, ..., X_n)$$

where X_1, X_2, . . . , X_n are the amounts of each of n goods X consumed. The marginal utility of X_1 is the extra utility obtained from slightly more X_1 while holding the amount of all other commodities constant.

The marginal utility of X_1 can be expressed by the function:

$$MU_{X_1} = \frac{\partial U}{\partial X_1}$$

G.1 Example: Find the marginal utility of X_1 and X_2 given $U = X_1^{\alpha} X_2^{\beta}$