Mathematics Applied to Quantitative Social Sciences Intermediate Level

Session 3

Juliana Pinillos
juliana.pinillos@sciencespo.fr

Sciences Po

September 29, 2020

Contents

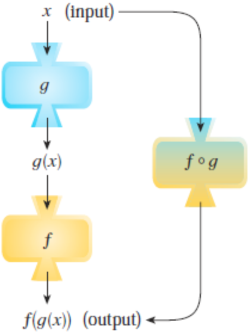
Composition of Functions

Composition of Functions

Let f and g be two functions. The composite function $f \circ g$ is defined by:

$$(f \circ g)(x) = f(g(x))$$

Domain $= DOM_{g \circ f} : x \in DOM_f \wedge f(x) \in DOM_g$



Example: Let $f(x) = x^3 + x$ and $g(x) = x + 2$

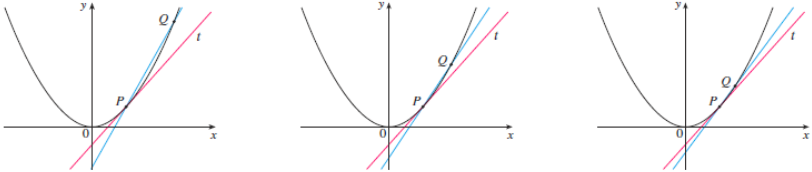
- $f \circ g = ?$
- $g \circ f = ?$

Composition of Functions

- $f \circ g = f(g(x))$
  $f \circ g = ((x+2)^3 + (x+2)$
  $f \circ g = (x^3 + 6x^2 + 12x + 8) + (x+2)$
  $f \circ g = x^3 + 6x^2 + 13x + 10$
- $g \circ f = g(f(x))$
  $g \circ f = (x^3 + x) + 2$
  $g \circ f = x^3 + x + 2$

The Tangent Problem

- The equation of the tangent line to the parabola $y = x^2$ at the point $P(1,1) \rightsquigarrow$ finding the slope m.
- $Q(x, x^2)$ is a nearby point on the parabola.
- Slope m_{PQ} of the secant line PQ : $mPQ = \frac{y_Q - y_P}{x_Q - x_P}$



The Tangent Problem

In the following slides, the exact definition of a limit is not required, it is enough to understand the limit in the context of the tangent problem.

x	m_{PQ}
2	3
1.5	2.5
1.1	2.1
1.01	2.01
1.001	2.001

x	m_{PQ}
0	1
0.5	1.5
0.9	1.9
0.99	1.99
0.999	1.999

$$m = \lim_{Q \to P} m_{PQ} = \lim_{x \to 1} \frac{x^2-1}{x-1}$$

Example:

A.1 What is the equation of the tangent line ?

Tangent Lines

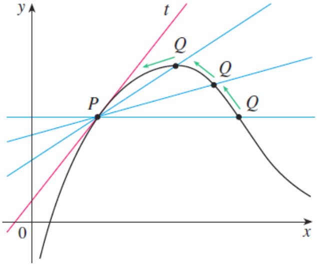
Interpretation of the derivative

Definition

The tangent line to the curve $y = f(x)$ at the point $P(a, f(a))$ is the line through P with slope:

$$m = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

Provided that this limit exists



Derivatives

Definition

The **derivative** of a function f at a number a, denoted by $f'(a)$, is

$$f'(a) = \lim_{x \to a} \frac{f(x)-f(a)}{x-a}$$

If this limit exists

The tangent line to $y = f(x)$ at $P(a, f(a))$ is the line through P whose slope is $f'(a)$

- Note that $\lim_{x \to a} \frac{f(x)-f(a)}{x-a} = \lim_{h \to 0} \frac{f(a+h)-f(a)}{h}$

B.1 **Example** : Find the derivative of $f(x) = x^2 - 8x + 9$

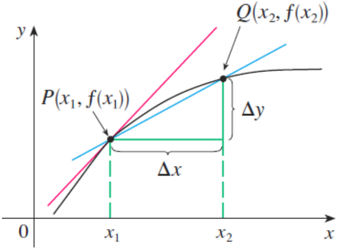
Rates of Change

Interpretation of the derivative

- y is a quantity that depends on another quantity $x : y = f(x)$
- If x changes from x_1 to x_2, the change (or increment) of x is $\Delta x = x_2 - x_1$
- The corresponding change of y is $\Delta y = f(x_2) - f(x_1)$
- The **average rate of change** of y with respect to x over $[x_1, x_2]$ is $\frac{\Delta y}{\Delta x}$
- The **instantaneous rate of change** of y with respect to x at x_1 is $\lim_{x \to 0} \frac{\Delta y}{\Delta x} = \lim_{x_2 \to x_1} \frac{f(x_2) - f(x_1)}{x_2 - x_1} = f'(x_1)$

Instantaneous Rate of Change

Interpretation of the derivative



- average rate of change from P to $Q = m_{PQ} =$ slope of the secant line (PQ)
- instanteneous rate of change at $P =$ slope of the tangent line at P

The derivative $f'(a)$ is the instantaneous rate of change of $y = f(x)$ with respect to x when $x = a$.

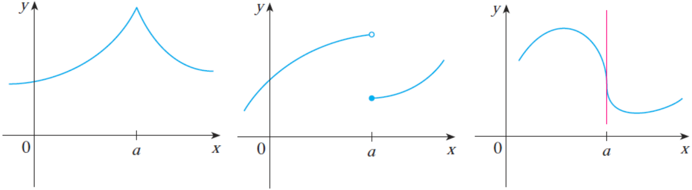
- **Example in economics:** marginal cost $=$ rate of change of production cost with respect to the number of items produced.

Differentiable Functions

Definition

- A function f is differentiable at a if $f'(a)$ exists $(f'(a) \in \mathbb{R})$
- A function f is differentiable on an open interval $[a, b]$ if it is differentiable at every point in the interval.

- $f(x) = |x|$ is not differentiable at 0
- $g(x) = \sqrt{x}$ is not differentiable at (the right of) 0
- Other examples of non differentiable functions:



Derivatives of Usual Functions

- Leibniz Notation:

$$f'(x) = \frac{d}{dx}(f(x))$$

- Derivative of a constant function

$$\frac{d}{dx}(c) = 0$$

- **The Power Rule :** if n is any real number,

$$\frac{d}{dx}(x^n) = nx^{n-1}$$

Derivatives of Usual Functions

- **The Constant Multiple Rule :** if c is a constant and f is a differentiable function, then:

$$\frac{d}{dx}(cf(x)) = (c)\frac{d}{dx}(f(x))$$

- **The Sum Rule:**

$$\frac{d}{dx}(f(x)+g(x)) = \frac{d}{dx}(f(x)) + \frac{d}{dx}(g(x))$$

- **The Difference Rule:**

$$\frac{d}{dx}(f(x)-g(x)) = \frac{d}{dx}(f(x)) - \frac{d}{dx}(g(x))$$

C.1 **Example :** find the points on the curve $y = x^4 - 6x^2 + 4$ where the tangent line is horizontal.

Derivatives of Usual Functions

The chain rule

- **The product rule:**

$$(f(x) \times g(x))' = f'(x)g(x) + f(x)g'(x)$$

- **The quotient rule:**

$$\left(\frac{f(x)}{g(x)}\right)' = \frac{f'(x)g(x) - f(x)g'(x)}{g(x)^2}$$

The Chain Rule

If g is differentiable at x and f is differentiable at $g(x)$, then the composite function $F = f \circ g$ is differentiable at x and

$$F'(x) = f'(g(x)) \times g'(x)$$

Differentiation Rules

Consequences of the Chain Rule

- **The power rule:**

$$\frac{d}{dx}(g(x))^n = ng'(x)(g(x))^{n-1}$$

- **Derivative of natural exponential functions**

$$\frac{d}{dx}(\mathrm{e}^x) = \mathrm{e}^x \qquad \frac{d}{dx}(\mathrm{e}^{g(x)}) = g'(x)\mathrm{e}^{g(x)}$$

- **Derivative of natural logarithmic functions:**

$$\frac{d}{dx}(\ln x) = \frac{1}{x} \qquad \frac{d}{dx}(\ln g(x)) = \frac{g'(x)}{g(x)}$$

- **Derivative of exponential functions:**

$$\frac{d}{dx}(a^x) = a^x \ln a$$

Differentiation Rules

- **Derivative of logarithmic functions:**

$$\frac{d}{dx}(\log_a x) = \frac{1}{x \ln a}$$

- **Example:** Find the derivative of
  D.1 $f(x) = \sqrt{1 + 2e^{5x}}$
  D.2 $f(x) = \dfrac{x}{\sqrt{x^2 + 1}}$
  D.3 $h(x) = 5^{-\frac{1}{x}}$
  D.4 $F(x) = \sqrt{x + \sqrt{x}}$

Marginal Revenue[1]

Having the average revenue of a firm (AR), as a function that depends on the product $AR = f(Q)$, it's possible to find it's marginal revenue (MR).
Example

$$AR = 15 - Q$$

We find the total revenue multiplying AR by Q

$$TR = Q(AR) = Q(15 - Q) = 15Q - Q^2$$

Differenciating TR,

$$MR = \frac{dTR}{dQ} = 15 - 2Q$$

Then the marginal revenue of the company it's given by $15 - 2Q$

[1]Examples taken from Alpha C. Chiang (1984) Fundamental Methods of Mathematical Economics.