Mathematics Applied to Quantitative Social Sciences Intermediate Level

Session 2

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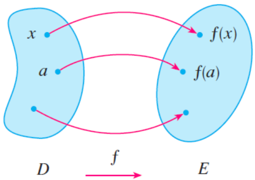
Outline

Functions

Definition

A function $f(x)$ is a rule that assigns to each element x in a set D **exactly one element**, called $f(x)$, in a set E.

- D is called the **domain** of the function
- $f(x)$ is the value of f at x
- The **range** of f is the set of all possible values of $f(x)$ as x varies throughout the domain, that is, the set $\{f(x), x \in D\}$.



Functions

- A symbol that represents a number in the domain of f is called an **independent variable**
- A symbol that represents a number in the range of f is called a **dependent variable**

Definition

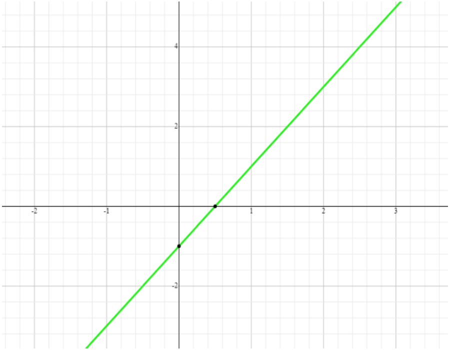
The graph of f is the set of ordered set pairs $\{(x, f(x)) \mid x \in D\}$

Examples

A.1) Graph domain and range of $f(x) = 2x - 1, g(x) = x^2$

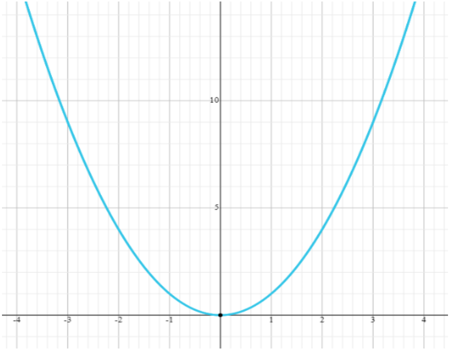
A.2) Domain of $f(x) = \frac{1}{\sqrt{x^2-3x+2}}$

Functions: $f(x) = 2x - 1$



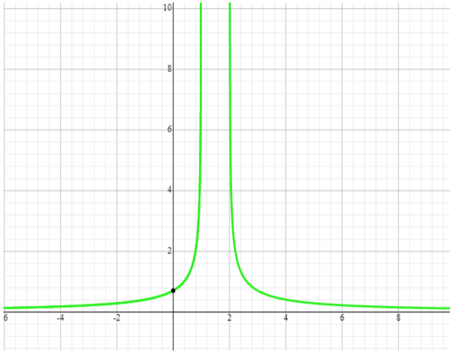
Domain: $(-\infty\ ,\ \infty\ )$
Range: (-∞ , ∞)

Functions: $g(x) = x^2$



Domain: $(-\infty\ ,\ \infty\ )$
Range:$[0,\ \infty\ )$

Functions: $f(x) = \frac{1}{\sqrt{(x^2-3x+2)}}$



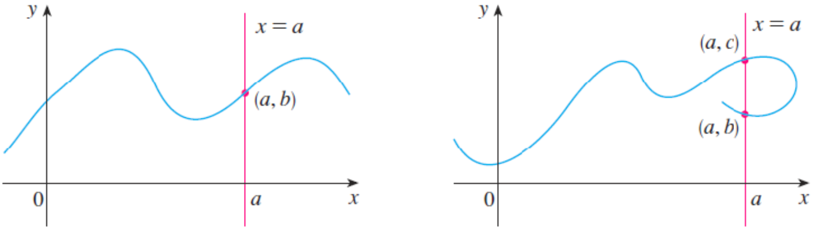
Domain: $(-\infty\ ,\ 1) \cup (2,\ \infty\ )$
Range: $(0,\ \infty\ )$

Functions

Vertical line test

Vertical line test

A curve in the xy-plane is the graph of a function of x if and only if no vertical line intersects the curve more than once.

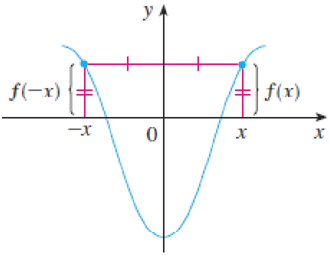


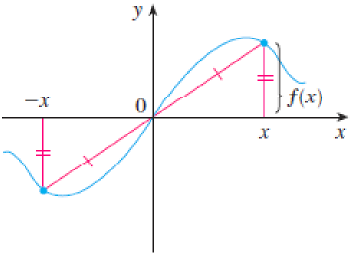
Functions

Symmetry

Symmetry

- If a function f satisfies $f(-x) = f(x)$ for every x in its domain, it is called an **even function**.
- If a function f satisfies $f(-x) = -f(x)$ for every x in its domain, it is called an **odd function**.





Examples

Determine whether each of the following functions is even, odd or neither

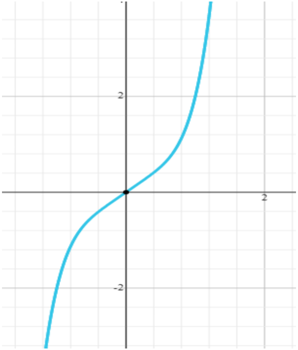
A.3) $f(x) = x^5 + x$

A.4) $g(x) = 1 - x^4$

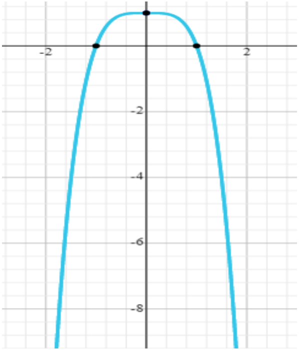
A.5) $h(x) = 2x - x^7$

Examples

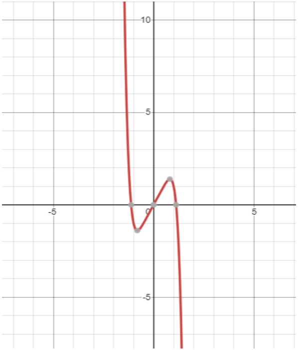
$f(x) = x^5 + x$



$g(x) = 1 - x^4$



$h(x) = 2x - x^7$

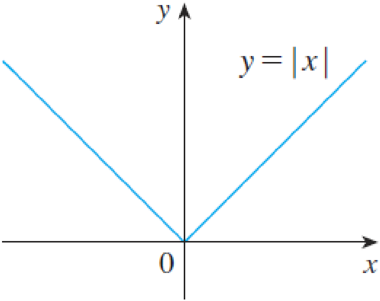


The Absolute Value Function

Let x be a real number. The **absolute value** of x is the distance from x to 0 on the real number line, denoted $|x|$.

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x \leq 0 \end{cases}$$

- Graph of $y = |x|$

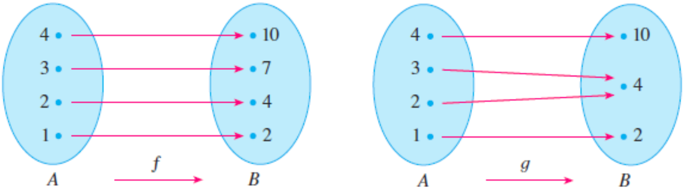


Inverse Functions

Definition

A function f is called a **one-to-one** function if it never takes on the same value twice ; that is

$$f(x_1) \neq f(x_2) \textbf{ whenever } x_1 \neq x_2.$$



B.1) Is f one-to-one?

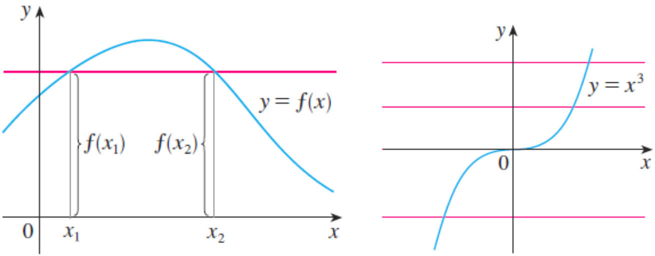
B.2) Is g one-to-one?

Inverse Fuctions

Horizontal line test

A function is **one-to-one** if and only if no horizontal line intersects its graph more than once.

Not to confuse with the vertical line test!

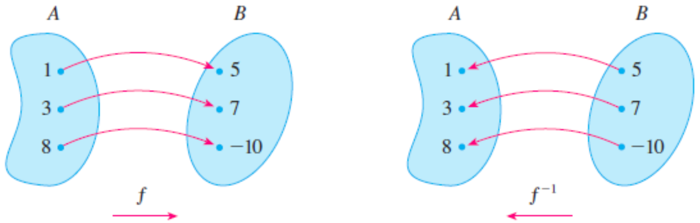


Inverse Functions

Definition

Let f be a one-to-one function with domain A and range B. Then its **inverse function** f^{-1} has domain B and range A and is defined by:

$$\forall\, y \in B, f^{-1}(y) = x \Leftrightarrow f(x) = y$$



$f^{-1}(x)$ **does <u>NOT</u> mean** $\frac{1}{f(x)}$

Inverse Functions

Cancellation equations

$$\forall x \in A,\ f^{-1}(f(x)) = x$$

$$\forall y \in B,\ f(f^{-1}(y)) = y$$

How to find the inverse function of a one-to-one function f

- Write $y = f(x)$
- Solve this equation for x in terms of y
- To express f^{-1} as a function of x, interchange x and y.

The graph of $f^{-1}$is obtained by reflecting the graph of f about the line $y = x$.

Example

B.3) Find the inverse function of $f(x) = x^3 + 2$

Linear Functions - Slope

- **Linear functions :** "y is a linear function of x" means that the graph of the function is a line. The function is of the form

$$y = f(x) = mx + b$$

  - m is the **slope** of the line
  - b is the y-**intercept**

Slope / Point-slope equation

- The **slope** of the line through $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ with $x_1 \neq x_2$ is:

$$\frac{y_2 - y_1}{x_2 - x_1}.$$

- The equation of the line through $P_1(x_1, y_1)$ and with slope m is:

$$y - y_1 = m(x - x_1)$$

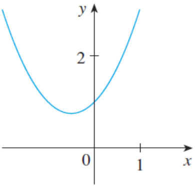
Polynomial Functions

Polynomial functions are of the form:

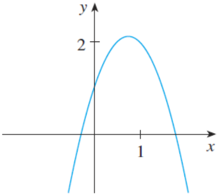
$$P(x) = a_n x^n + a_{n-1} x^{n-1} + ... + a_1 x + a_0$$

where $n \geq 0$, the a_i's are constants called the coefficients of the polynomial.

- Domain $= \mathbb{R}$
- If $a_n \neq 0$, the degree of P is n.
- example : quadratic functions $ax^2 + bx + c$: parabola.



(a) $y = x^2 + x + 1$



(b) $y = -2x^2 + 3x + 1$

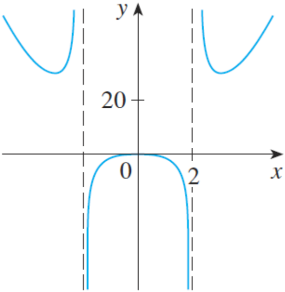
Rational Functions

Rational functions are ratio of two polynomials

$$f(x) = \frac{P(x)}{Q(x)}$$

Domain $= \{x \in \mathbb{R}, Q(x) \neq 0\}$

Example: $f(x) = \frac{2x^4 - x^2 + 1}{x^2 - 4}$

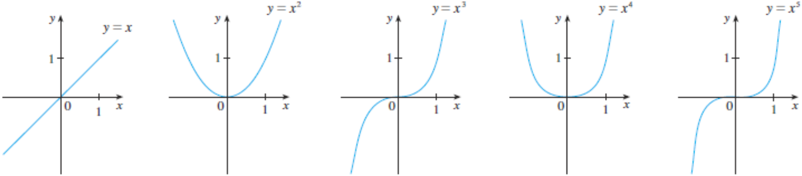


Power Functions

Power Functions:

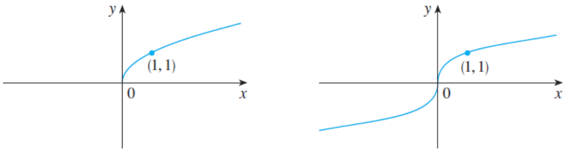
$$f(x) = x^a, a \text{ is a constant.}$$

- $a = n$ a positive integer



Power Functions

- $a = \frac{1}{n}$, n a positive integer: $f(x) = x^{\frac{1}{n}} = \sqrt[n]{x}$ root function.



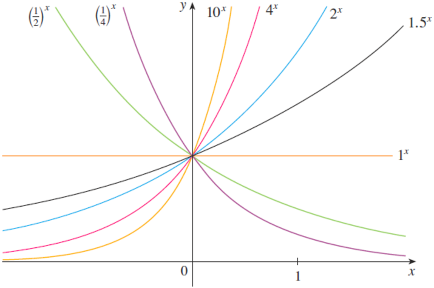
(a) $f(x) = \sqrt{x}$ (b) $f(x) = \sqrt[3]{x}$

- $a = -1$: $f(x) = x^{-1}$ the reciprocal function

Exponential Functions

$$f(x) = a^x, a = \text{positive constant}$$

- **Not to confuse with Power functions**
- domain $= \mathbb{R}$
- Range $= [0, +\infty]$



Exponential Functions

Application of exponential functions occur frequently in mathematical models of nature and society.

Example: A population of bacteria which doubles in size every hour. Let $p(t) =$ number of bacteria at time t measured in hours.

- Write the function in terms of t

Exponential Functions

Application of exponential functions occur frequently in mathematical models of nature and society.

Example: A population of bacteria which doubles in size every hour. Let $p(t)$ = number of bacteria at time t measured in hours.

- Write the function in terms of t

Solution:

$$p(t) = 2^t$$

Exponential Functions

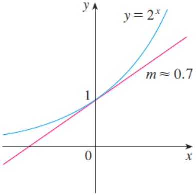
Laws of exponents

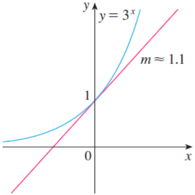
For a, b positive numbers, x, y real numbers.

- $a^{x+y} = a^x a^y$
- $a^{x-y} = \frac{a^x}{a^y}$
- $(a^x)^y = a^{xy}$
- $(ab)^x = a^x b^x$

Natural Exponential Function

the number e





$m =$ the slope of the **tangent line** at $(0,1)$ for $f(x) = a^x$.

The number e

The purpose of choosing e is to make m equal to 1.

$$e \approx 2.71828$$

$f(x) = e^x$ is the **natural exponential function**

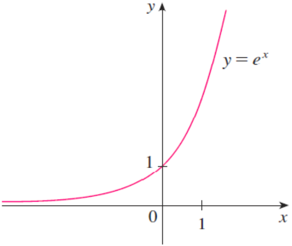
Natural Exponential Function

To sum up

The natural exponential function $f(x) = e^x$ is the exponential function such that:

$$\lim_{x\to 0} \frac{f(x)-f(0)}{x-0} = \lim_{x\to 0} \frac{e^x-1}{x} = 1$$

- $f(x) = e^x$ is defined, continuous, strictly positive and differentiable on $\mathbb{R}$.



Natural Exponential Function

- $f(x) = \mathrm{e}^x$ is one-to-one.

$$\forall x, y \in \mathbb{R}, (\mathrm{e}^x = \mathrm{e}^y) \Leftrightarrow (x = y)$$

- $f(x) = \mathrm{e}^x$ is strictly increasing on $\mathbb{R}$

$$\forall x, y \in \mathbb{R}, (\mathrm{e}^x < \mathrm{e}^y) \Leftrightarrow (x < y)$$

Laws of exponents

For x, y real numbers:

- $\mathrm{e}^{x+y} = \mathrm{e}^x \mathrm{e}^y$
- $(\mathrm{e}^x)^y = \mathrm{e}^{xy}$
- $\mathrm{e}^0 = 1$
- $\mathrm{e}^{x-y} = \frac{\mathrm{e}^x}{\mathrm{e}^y}$

Logarithmic Functions

- If $a > 0$ and $a \neq 1$, $f(x) = a^x$ is a one-to-one function.
- Its inverse function f^{-1} is called the **logarithmic function** with base a, denoted by $\log_a$

$$\boxed{\log_a x = y \Leftrightarrow a^y = x}$$

Hence, if $x > 0$, $\log_a x$ is the exponent to which the base a must be raised to give x.

Cancellation equations

$$\forall x \in \mathbb{R}, \log_a(a^x) = x$$

$$\forall x > 0, a^{\log_a x} = x$$

C.1) $\log_{10}(0.001) = ?$

C.2) $\log_2(32) = ?$

C.3) $\log_5(0.2) = ?$

Logarithmic Functions

Laws of logarithms

For x, y positive numbers

- $\log_a(xy) = \log_a x + \log_a y$
- $log_a\left(\frac{x}{y}\right) = log_a x - log_a y$
- $\log_a(x^\gamma) = \gamma \log_a(x)$

C.4 $\log_2(80) - log_2(5) = ?$

C.5 $\log_{10}(50) + \log_{10}(200) = ?$

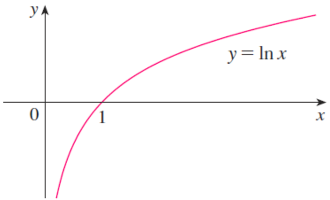
Natural Logarithmic Function

The inverse function of $f(x) = \mathrm{e}^x$ is called the **natural logarithmic function**, denoted by ln.

$$\boxed{\ln(x) = y \Leftrightarrow \mathrm{e}^y = x}$$

$g(x) = \ln(x)$ is defined, continuous and differentiable on $\mathbb{R}_+^*$

- Graph of $g(x) = \ln(x)$



Natural Logaritmic Function

Cancellation equations

$$\forall x \in \mathbb{R}, \ln(\mathrm{e}^x) = x$$

$$\forall x > 0, \mathrm{e}^{\ln \mathrm{e}^x} = x$$

- $g(x) = \ln(x)$ is one-to-one.

$$\forall x, y \in \mathbb{R}_+^*, (\ln(x) = \ln(y)) \Leftrightarrow (x = y)$$

- $\mathrm{g(x)} = \ln(\mathrm{x})$ is strictly increasing on $\mathbb{R}_+^*$

$$\forall x, y \in \mathbb{R}_+^*, (x < y) \Leftrightarrow (\ln(x) < \ln(y))$$

Natural Logaritmic Function

Laws of natural logarithm

For $x,\ y > 0$

- $\ln(xy) = \ln(x) + \ln(y)$
- $\ln\left(\frac{x}{y}\right) = \ln(x) - \ln(y)$
- $\forall r \in \mathbb{R}, \ln(x^r) = r\ln(x)$
- $\ln(1) = 0$
- $\ln\left(\frac{1}{x}\right) = -\ln(x)$

D.1) $\ln(\mathrm{e}) = ?$

D.2) Solve $\ln(x) = 5$

D.3) Solve $\mathrm{e}^{5-3x} = 10$

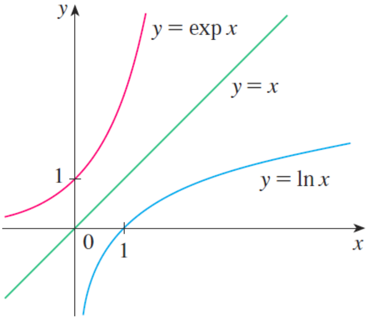
Logarithms with any base can be expressed in terms of the natural logarithm, thanks to :

Change of base formula

For any positive number a $(a \neq 1)$:

$$\log_a x = \frac{\ln x}{\ln a}$$

Natural Exponential and Logarithmic Functions



Examples - Application

Figure 1: World

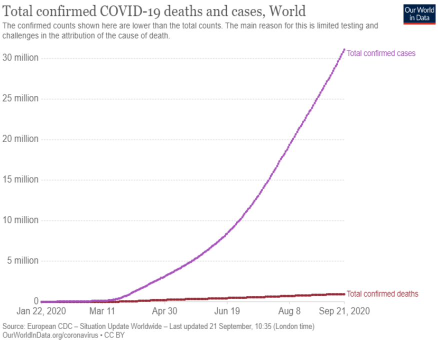
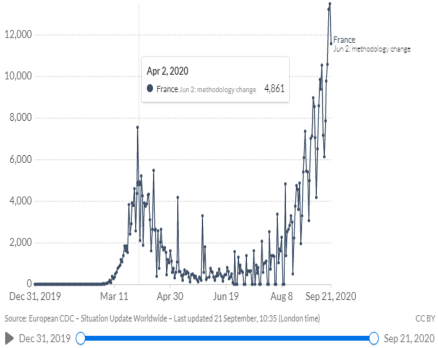
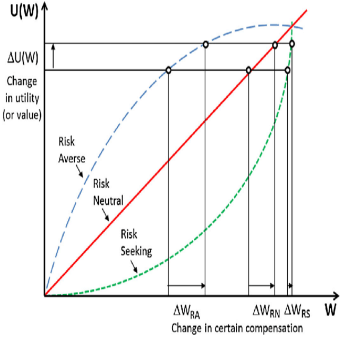


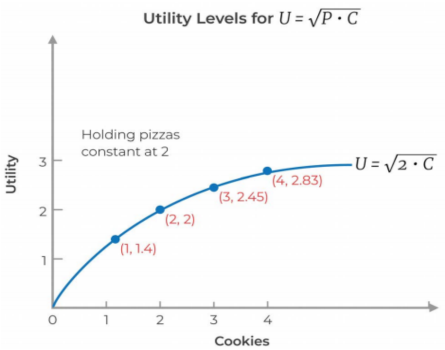
Figure 2: France



Note: Our World in Data (2020)

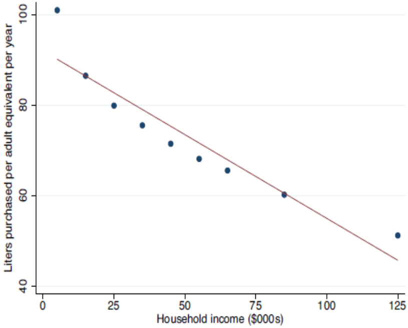
Examples - Utility Functions

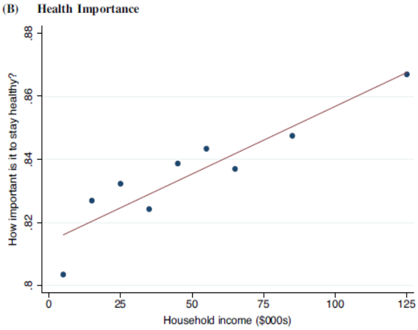




Source: Gruber (2018). MIT Lecture Notes

Examples - Linear Functions





Source: Allcot, Lockwood, Taubinsky (2017)