Mathematics Applied to Quantitative Social Sciences Intermediate Level

Session 1

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September 15, 2020

Contents

Fractions

A fraction is a number written as $\frac{a}{b}$ where a and b are both integers and $a \neq b$

- Addition/Substraction:

$$\frac{a}{b} \pm \frac{c}{d} = \frac{ad \pm bc}{bd}$$

- Product:

$$\frac{a}{b} \times \frac{c}{d} = \frac{ac}{bd}$$

- Quotient:

$$\frac{\frac{a}{b}}{\frac{c}{d}} = \frac{a}{b} \times \frac{d}{c} = \frac{ad}{bc}$$

- Simplifying:

$$\frac{a \times k}{b \times k} = \frac{a}{b}$$

Expansion/Factorization

Let a, b, c be real numbers

- $a(b \pm c) = ab \pm ac$
- $(a+b)^2 = a^2 + 2ab + b^2$
- $(a-b)^2 = a^2 - 2ab + b^2$
- $(a-b)(a+b) = a^2 - b^2$
- $(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$
- $(a-b)^3 = a^3 - 3a^2b + 3ab^2 - b^3$

Expansion/Factorization

$ax^2 + bx + c$

Consider the polynomial $ax^2 + bx + c$ (with $a \neq 0$) and let $\triangle = b^2 - 4ac$ be its discriminant.

- If $\triangle < 0$, the polynomial cannot be factorized.
- If $\triangle > 0$, the **roots** of the polynomial are:

$$x_1 = \frac{-b + \sqrt{\triangle}}{2a}, x_2 = \frac{-b - \sqrt{\triangle}}{2a}$$

And the polynomial can be factorized as following:

$$\boxed{ax^2 + bx + c = a(x - x_1)(x - x_2)}$$

It may be useful to notice that

$$x_1 + x_2 = -\frac{b}{a}, x_1 x_2 = \frac{c}{a}$$

Powers

Let a be non zero and n be a positive integer, then a^n is defined to be:

$$a^n = \underbrace{a \times a \times \ldots \times a}_{n \text{ times}}$$

and a^{-n} is defined to be:

$$a^{-n} = \frac{1}{a^n}$$

Power rules

Let a, b be non zero numbers and n, p be integers, then:

- $(ab)^n = a^n b^n$
- $a^{n+p} = a^n a^p$
- $a^{n-p} = \frac{a^n}{a^p}$
- $(a^n)^p = a^{np}$
- $a^0 = 1$

Examples

Simplify the following:

a) $(\frac{3}{4} \times \frac{5}{7}) \div \frac{3}{2}$

b) $\frac{3}{4} - \frac{1}{2} \div \frac{13}{2}$

c) $\frac{10^{-8} \times 0.7 \times 10^{12}}{21 \times 10^{3}}$

Factorize:

d) $x^2 - 5x + 6$

e) $2p^3 - p^2 + 2p - 1$

f) $x^4 - y^4$

Sol.

Percent Change

Suppose a quantity has an initial value of x_1 and then increases or decreases to a final value of x_2. The percent change $t\%$ is calculated as follows:

$$t\% = \frac{x_2 - x_1}{x_1} \times 100$$

- If $x_2 > x_1$, then t is a positive number $\rightsquigarrow$ increase.
- If $x_2 < x_1$, then t is a negative number $\rightsquigarrow$ decrease.

Examples [1]

- A pair of pants costs $24. The cost was reduced by 8%. What is the new cost of the pants?

[1] Examples taken from (McKibben, 2018).

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- Kyra receives a 5% commission on every car she sells. She received a \$1,325 commission on the last car she sold. What was the price of the car?

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- Kyra receives a 5% commission on every car she sells. She received a \$1,325 commission on the last car she sold. What was the price of the car? $\rightarrow 1325 = p \cdot (0.05) \rightarrow p = 26500$

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Absolute and Relative Values

- **Absolute value:** a precise real number. Ex.: 3 workers, 5000 USD.
- **Relative value:** It is a number whose magnitude and interpretation depends on another number or value.

Absolute and Relative Values: Econ. Application

- **Absolute poverty:** The amount of money to cover "basic necessities of life" could be \$ 200 in Colombia. If a household's income is below that, is classified as poor.
- **Relative poverty:** When households receive a given percentage (50%) less than of the median income. For instance, is the Colombian median income is \$ 600, and you earn \$250, you are "relative poor", as you earn less than \$ 300 (50% of the median). Notice you are not an absolute poor household.

Exact and Approaches Values

- Some times we want to be exactly precise about one quantity, we call this an **“exact value”**. For instance:
  - the FED raised the interest rate from 1.25% up to 1.75 %;
  - the Bank pays you a real interest rate of 6.85%.

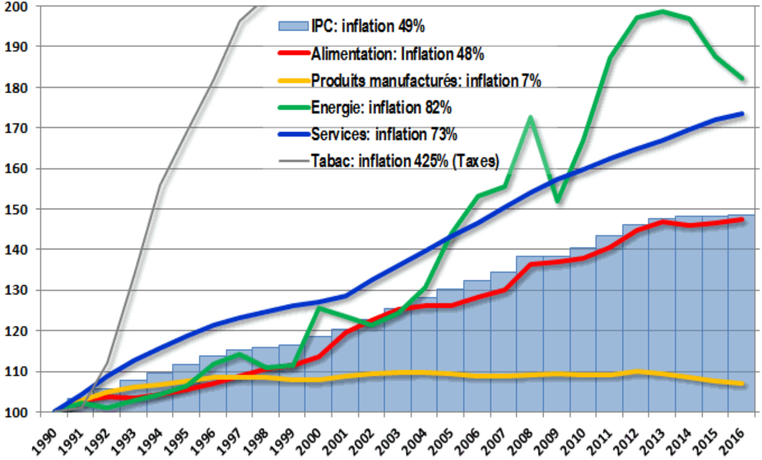
  In both cases, precision **DOES** matter: is not 2% or 7%! as this could be a lot money if you have thousand of millions to invest.

- In other occasions, we want to be less rigorous. We would prefer an **approach value**. Usually .5 is a cutoff. Above this, we approximate to the closest higher integer. For instance:
  - The marginal tax rate that richest household pay is 33%. Easier for accounting than setting up 32.8%.
  - Some professors like to round up grades: 14.75 $\simeq$15.0

Index

Some times is useful to use an index in order to make comparisons. The number per se does not have an interpretation, but it is magnitude help us to understand the evolution of a variable.

Example: The Consumer Price Index (CPI) could take the value of 100 for a baseline-year, say 2010. If in 2015, CPI is 120, it means that prices have gone up 20%.



200
190
180
170
160
150
140
130
120
110
100
IPC: inflation 49%
Alimentation: Inflation 48%
Produits manufacturés: inflation 7%
Energie: inflation 82%
Services: inflation 73%
Tabac: inflation 425% (Taxes)
1990 1991 1992 1993 1994 1995 1996 1997 1998 1999 2000 2001 2002 2003 2004 2005 2006 2007 2008 2009 2010 2011 2012 2013 2014 2015 2016

Ingredients of an Economic Model

- **Variable:** something whose magnitude change. Variables that are determined by the system are *endogenous* (quantities, profits, etc). Variables that are given are known as *exogenous* (international prices, weather, etc.).
- **Constant:** is a magnitude that does not change. It is the antithesis of a variable.
- **Parameter:** a constant joined to a variable. Also called the coefficient. It's scaling the value of the variable.

Linear systems with two variables

- A linear system with two equations of two variables x and y is any system that can be written in the form:

$$\begin{cases} ax + by = \alpha \\ cx + dy = \beta \end{cases}$$

where a, b, c, d, α and β are constants.

- A solution to the system is a value of x and a value of y that satisfy both equations at the same time.
- A system can have:
  - Exactly one solution.
  - Infinitely many solutions.
  - No solutions.

Linear Systems with Two Variables

Example [2]

- A company has a 100-hectares farm on which it grows lettuce and corn. Each hectare (ha) of corn requires 600 hours of labor, and each ha of lettuce needs 400 hours of labor. If 45,000 hours are available and if all land and labor resources are to be used, find the number of has of each crop that should be planted.

Sol.

[2] Examples taken from (Sworowski and Cole, 2010).

Substitution Method

1. Solve one of the equations for one of the variables, say x.
2. Substitute x into the other equation.
3. This yields one equation with the other variable y, that we solve.
4. Substitute the other value of y back into one of the equations.
5. Find the value of the remaining variable x.

Example:

$$\begin{cases} 2x - 3y = -2 \\ 4x + y = 24 \end{cases}$$

Substitution Method - Example

1. Solve the first equation for the variable x.

$$x = \frac{3}{2}y - 1$$

2. Substitute x into the other equation.

$$4(\frac{3}{2}y - 1) + y = 24$$

3. This yields one equation with the other variable y, that we solve.

$$\boxed{y = 4}$$

4. Substitute the other value of y back into one of the equations.

$$4x + 4 = 24$$

5. Find the value of the remaining variable x.

$$\boxed{x = 5}$$

Elimination Method

1. Multiply one or both equations by appropriate numbers so one of the variables will have the same coefficient with opposite signs.
2. Add the two obtained equations together. One of the variables will be eliminated.
3. The result is a single equation that we can solve for one of the variables.
4. Substitute this answer back into one of the original equations.

Example:

$$\begin{cases} 5z + 4y = 1 \\ 3z - 6y = 2 \end{cases}$$

Elimination Method - Example

1. Multiply one or both equations by appropriate numbers so one of the variables will have the same coefficient with opposite signs.

$$5z + 4y = 1 \xrightarrow{\times 3} 15z + 12y = 3$$
$$3z - 6y = 2 \xrightarrow{\times 2} 6z - 12y = 4$$

2. Add the two obtained equations together. One of the variables will be eliminated.

$$21z = 7$$

3. The result is a single equation that we can solve for one of the variables.

$$\boxed{z = \frac{1}{3}}$$

4. Substitute this answer back into one of the original equations.

$$3\left(\frac{1}{3}\right) - 6y = 2 \Rightarrow \boxed{y = -\frac{1}{6}}$$

Applications: Supply and Demand

- The quantity of a product that people are willing to buy depends on its price. Generally, the higher the price, the less the demand.
  $\rightsquigarrow$ **Demand equation**
- Similarly, the quantity of a product that a supplier is willing to sell also depends on the price. Generally, a supplier will be willing to supply more of a product at higher prices and less of a product at lower prices.
  $\rightsquigarrow$ **Supply equation**
- The simplest supply and demand model is a linear model.

$$p = aq + b \qquad \textbf{(Demand equation)}$$

$$p = cq + d \qquad \textbf{(Supply equation)}$$

Where $p =$ price and $q =$ quantity.

Applications: Supply and Demand

Example

- At what price will supply equal demand ? This price, if it exists, is called the **equilibrium price**, and the quantity sold at the price is called the **equilibrium quantity.**
- To find the equilibrium price and equilibrium quantity, we solve the linear system (with variables p and q) :

$$\begin{cases} p = aq + b \\ p = cq + d \end{cases}$$

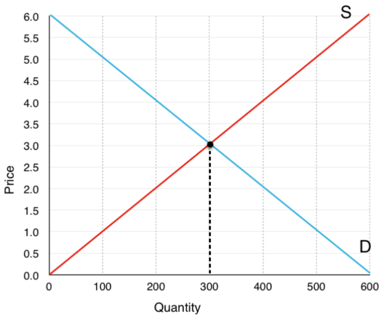
Example: The price demand and price supply equations for strawberry in a certain country are:

$$p = -0.2q + 4 \qquad \textbf{(Demand equation)}$$

$$p = 0.04q + 1.84 \qquad \textbf{(Supply equation)}$$

where q = quantity in thousands of pounds and p = price in dollars. Find the **equilibrium price** and the **equilibrium quantity**.

A simple model - Demand and Supply curves



Equilibrium price = 3$
Equilibrium quantity = 300 units

Additional definitions

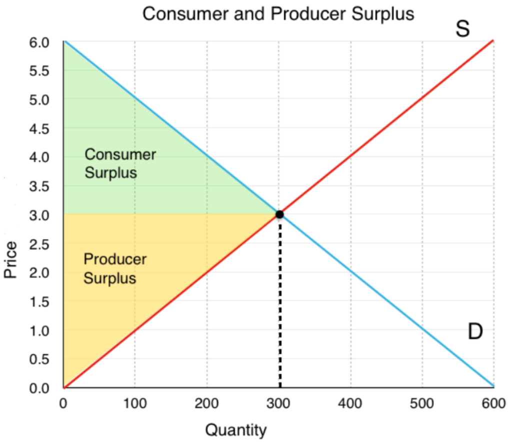
Consumer and producer surplus at equilibrium

Definition

- **Consumer surplus** = the difference between the consumers' willingness to pay for a commodity and the equilibrium price.
- **Producer surplus** = the difference between the amount the producer is willing to supply goods for and the equilibrium price.

- How to calculate consumer and producer surplus as areas ?

Consumer and producer surplus as areas



Solution: Percent Change

a) $5/14$

b) $1/26$

c) $1/3$

d) $(x-3)(x-2)$

e) $(2p-1)(p^2+1)$

f) $\left(x^2+y^2\right)(x+y)(x-y)$

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Solution: Linear Systems with Two Variables

We have the following system:

$$c + let = 100$$
$$600c + 400let = 45000$$

Where c stands for corn and *let* for lettuce.
We know that $c = 100 - let$. We substitute this in the second equation:

$$600(100 - let) + 400let = 45000$$
$$60000 - 600let + 400let = 45000$$
$$let = 75$$

Then, $c = 25$.

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References

McKibben, M. A. (2018). *501 math word problems*. Learning Express.

Sworowski, E., & Cole, J. A. (2010). *Algebra and trigonometry with analytic geometry* (12th). Brooks/Cole, Cengage Learning.